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Kantorovich and Zalgaller (1951): the 0-th column generation algorithm

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Abstract

This article probes the origins of the Column Generation technique. It begins with Kantorovich's classic 1939 work, correcting widespread misconceptions about his contributions to the Cutting Stock Problem. It then brings to light Kantorovich and Zalgaller's lesser-known 1951 book, which is revealed to contain a complete Column Generation algorithm. The article also places these contributions in the context of the turbulent USSR's political and ideological environment, essential for a deeper understanding of their significance.

Keywords Column Generation · Linear Programming · Dynamic Programming · Cutting Stock Problem · History of Science

Mathematics Subject Classification (2020): Primary 90C05; Secondary 90C06 · 90C27 · 01A60

1 Introduction

Column Generation (CG) is a technique to solve Linear Programs (LPs) with a very large number of variables. It operates on a restricted LP, where the vast majority of those variables are omitted. Then, auxiliary optimization problems, known as pricing subproblems, implicitly evaluate the reduced costs of the missing variables. While promising variables are identified, they are added to the restricted LP (the corresponding matrix *columns* are *generated*, hence the technique's name). CG is one of the major optimization techniques, being especially effective in integer programming when embedded in algorithms like Branch-and-Price and Branch-Cut-and-Price. It

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has been successfully applied to many types of problems, such as vehicle routing, cutting/packing, airline planning, timetabling, crew scheduling, graph coloring, clustering, lot sizing, and machine scheduling. CG has also found its way into industry, where it is routinely used (usually as part of highly effective heuristics, optimality being a secondary concern) for handling complex optimization problems where many millions of dollars are at stake.

The CG literature (e.g., [9, 38, 41]) recognizes the following works as those that introduced that technique:

1. The precursor work by Ford Jr. and Fulkerson [15]. They proposed an early CG algorithm for the following maximum multi-commodity flow problem: given a graph $G = (V, A)$ with arc capacities u_a, $a \in A$, and K commodities, commodity $k \in [K] = \{1, \dots, K\}$ having S^k and T^k as their source and sink sets, respectively; find a maximum total flow. They formulated the problem as:

$$\max \quad \sum_{k\in[K]} \sum_{\boldsymbol{p}\in\Omega^k} \lambda^k_{\boldsymbol{p}} \tag{1a}$$

$$\text{s.t.} \quad \sum_{k\in[K]} \sum_{\boldsymbol{p}\in\Omega^k} \boldsymbol{p}_a \, \lambda^k_{\boldsymbol{p}} \le u_a \qquad a \in A \tag{1b}$$

$$\boldsymbol{\lambda} \ge 0, \tag{1c}$$

   where Ω^k is the set containing the incidence vectors of all paths that start at some vertex in S^k and end at some vertex in T^k; for a vector $\boldsymbol{p} \in \Omega^k$, $\boldsymbol{p}_a$ indicates whether an arc $a \in A$ belongs to the corresponding path and $\lambda^k_{\boldsymbol{p}}$ is the flow of commodity k carried over that path. Despite the huge number of variables, they realized that these LPs could be solved by the Revised Simplex Algorithm [6], using the shortest path algorithm to perform the pricing step.
   Their main motivation for the CG approach was not reducing computing time. They noticed that large maximum multi-commodity flow problems could not be handled as standard LPs by the Simplex method since their base matrices would not even fit in the main memory of the computers then available. They mentioned a hypothetical instance with 50 vertices, 100 arcs, and 20 commodities. In that case, there would be 1000 flow conservation constraints and 100 arc capacity constraints, so a base matrix would have dimension 1100×1100. On the other hand, the proposed LP with path variables would have base matrices of dimension 100×100. Their article concludes with: “Except for hand computation for a few small problems, we have no computational experience with the proposed method. Whether the method is practicable [...] is a question that can only be settled by experimentation.”
2. The fundamental work by Dantzig and Wolfe [7] proposing a general technique for reformulating an LP that leads to fewer constraints but a potentially huge number of variables. Nevertheless, it was shown that the reformulated problem could be solved by the Revised Simplex Algorithm, using an auxiliary LP to perform the pricing step. The article discusses cases, including the one where the original constraint matrix having a block-diagonal structure, in which the pricing subproblem can be decomposed into several independent smaller LPs. The article also does not

provide computational experiments, only pointing out that in those cases it “holds promise for the efficient computation of large-scale systems”. It also discusses the economic and game-theoretical implications of the Dantzig-Wolfe decomposition: it is possible to obtain global optimal decisions in a system where a central planning agency (the Master LP) only communicates with a set of autonomous agents (the subproblems) by iteratively setting prices for shared resources (the dual variables) and receiving optimal production offers (the columns).

3. The works by Gilmore and Gomory [17, 18] on the Cutting Stock Problem (CSP). The considered CSP variant can be defined as follows: given J items with lengths w_j and demands (required number of copies) d_j, $j \in [J]$, and K stock types with standard lengths W_k and costs c_k, $k \in [K]$, find a way of producing the demand for each item that minimizes the cost of the used stocks. [17] presents a formulation based on the concept of *cutting patterns*, which are the essential ways of cutting a single stock, characterized by how many copies of each item are produced:

$$\min \sum_{k\in[K]} \sum_{\boldsymbol{q}\in Q^k} c_k \lambda^k_{\boldsymbol{q}} \tag{2a}$$

$$\text{s.t.} \quad \sum_{k\in[K]} \sum_{\boldsymbol{q}\in Q^k} \boldsymbol{q}_j \lambda^k_{\boldsymbol{q}} \geq d_j \qquad j \in [J] \tag{2b}$$

$$\boldsymbol{\lambda} \geq \mathbf{0} \text{ and integer,} \tag{2c}$$

where Q^k is the set of all cutting patterns for stock type k; for a vector $\boldsymbol{q} \in Q^k$, $\boldsymbol{q}_j$ indicates how many copies of item $j \in [J]$ are produced by the corresponding cutting pattern and $\lambda^k_{\boldsymbol{q}}$ is the number of stocks that are cut in that way. The potential number of cutting patterns can be very large. It was realized that the LPs obtained by dropping the integrality constraint could be solved efficiently by a CG algorithm where the pricing solves Integer Knapsack problems. Integer CSP solutions of excellent quality could then be obtained by rounding up fractional variables to the next integer or by rounding them down and treating the unfilled demand by ad hoc methods. A more advanced version of that method, which was already being used in the routine operation of a large paper mill, appears in [18]. That version proposes alternative methods for solving the knapsack subproblems and even includes several other practical constraints, such as limits on the number of knives available for cutting the paper rolls. Extensive computational results are presented and discussed, including the effects of having a larger/smaller stock size or multiple stock types on the waste. [19] considers the cutting of 2D rectangular stocks.

The goal of the present article is to bring additional information on the origins of CG through an in-depth analysis of parts of the following works:

– The booklet by Kantorovich [20], first published in Russian and later translated into English [26] as “Mathematical Methods of Organizing and Planning Production”. This is unquestionably one of the most important works in the history of Operations Research, presenting LP models for nine families of practical problems as well as an algorithm for their solution. His pioneering use of linear programming would

be recognized with the 1975 Nobel Memorial Prize in Economic Sciences, shared with the Dutch-American mathematician and economist Tjalling Koopmans [1]. Even though [26] is nowadays only a few clicks away from potential readers, the content of its chapter on the CSP are consistently misrepresented in the literature. We aim to correct those mistakes by describing in detail what is really written in that chapter.
- The book by Kantorovich and Zalgaller [21], published in Russian and whose title can be translated as "Rational Cutting of Industrial Materials".
  It is an extensive and mature treatment of the CSP, including its 2D variants, reflecting years of practical application of the presented methods. We intend to present one of those methods, which is a complete CG algorithm.

However, our article also has another dimension. It is not possible to properly explain those developments without mentioning some historical facts that had a profound impact on them. Therefore, we will try to contextualize those works in the context of the Stalinist USSR.

2 The CSP models in Kantorovich (1939)

Leonid V. Kantorovich (1912–1986) was a math prodigy, publishing his first papers at the age of 15. In 1934, at 22, he was already a full professor at the prestigious Leningrad (now Saint Petersburg) University. In 1938, he accepted a task to increase the output of a nearby plywood (material that consists of several thin layers of wood glued together) plant. It was a highly fruitful experience. Kantorovich realized that he could use his mathematics to solve to optimally a variety of production planning problems, as presented in [20]. The dense booklet (67 pages in the original, 57 pages in its English translation) has nine chapters, each proposing models (which we would now call LP models) for some family of problems, mostly in the industry but also in agriculture and transportation. The booklet also has three appendices. Appendix 1 presents his Method of Resolving Multipliers (MRM), which can be viewed as a Lagrangian method: certain constraints are dualized, their multipliers are adjusted until a near-optimal dual solution is found, and then a primal solution is recovered. Appendix 2 provides a detailed numerical solution by the MRM of the original problem posed by the plywood plant: the optimal time allocation of seven heterogeneous peeling machines (actually, there were eight machines but two of them are identical and can be aggregated) for producing five kinds of materials. It corresponds to an LP with 36 variables and 12 constraints. Up to that point, the text, aimed at economists, engineers, and managers, only uses relatively simple mathematics. Then, Appendix 3, titled "Theoretical Supplement", gives analytical and geometrical proofs of the existence of optimal resolving multipliers. It should be noted that the presented MRM was intended to solve the proposed models, which he classified into Problems A, B, and C. There are modifications in the multiplier adjusting procedure depending on the structure of each of those problems. The overall MRM was not developed for solving LPs as they

[1] Many people think that George B. Dantzig, creator of the highly impactful Simplex algorithm (1947), should also have been one of the recipients of that prize.

would be later defined by Dantzig. Yet, [29] showed that any LP can be transformed into a Problem C and solved by the MRM [2].

Chapter IV of [20], titled "Minimization of Scrap", deals with the CSP. A widespread error in the recent literature is attributing the so-called weak CSP formulation to it. We present that formulation for the case of a single stock type with length W. Let U be an upper bound on the number of stocks that need to be cut. This bound can be produced by any heuristic. Then, let binary variable y^u, $u \in [U]$, indicate whether stock u is indeed used in an optimal solution. Integer variables x_j^u, $j \in [J]$, $u \in [U]$, represent the number of copies of item j that are cut from stock u. The formulation is:

$$\min \quad \sum_{u \in [U]} y^u \tag{3a}$$

$$\text{s.t.} \quad \sum_{u \in [U]} x_j^u = d_j \qquad j \in [J] \tag{3b}$$

$$\sum_{j \in [J]} w_j x_j^u \le W y^u \qquad u \in [U] \tag{3c}$$

$$x_j^u \ge 0 \qquad j \in [J], u \in [U] \tag{3d}$$

$$0 \le y^u \le 1 \qquad u \in [U] \tag{3e}$$

$$\boldsymbol{x}, \boldsymbol{y} \text{ integer.} \tag{3f}$$

The weak CSP formulation is indeed poor. First, the optimal solution of its linear relaxation always yields the trivial lower bound $\sum_{j \in [J]} w_j d_j / W$. Second, its extreme symmetry makes branching and cutting highly ineffective. Third, it is not even polynomially sized, since there are classes of instances where U grows exponentially with the instance size. For example, it suffices to increase the item demands while keeping the remaining data fixed to obtain such a class.

The weak formulation seems to have been first proposed in [12] for the Bin Packing Problem, which is the particular case of the CSP where all demands are unitary. It reappears in [32]. It was shown in [36, 37] that the weak formulation could be used for deriving the strong Gilmore-Gomory formulation via Dantzig-Wolfe decomposition. After the 2000s, almost all authors (e.g., [8, 31, 40]) incorrectly attribute the weak formulation to Kantorovich. We ourselves also did that in our courses, repeating second-hand information. Only in 2022, when we read all the classic literature on the CSP (as part of the research for a forthcoming book on Column Generation), we could not find any trace of the weak formulation neither in that original work nor in its English translation. So, which CSP model is proposed in Chapter IV of [20]?

After verifying that [26] is an unabridged and accurate translation of [20] (at least for the parts relevant to this article), we make our detailed references to the English version, which is accessible to a broader international audience. The main Kantorovich CSP model (page 380) corresponds to the following variant. Suppose that a factory produces a certain article. Each article requires d_j units of item j, $j \in [J]$. There are K

[2] That statement was contested in [5], which led to a "Cold War LP controversy" [30]. More recent scholarship considers the objections by Charnes and Cooper as technicalities and accepts the MRM as a general LP method [35]. Yet, the LP format introduced by Dantzig is definitely more intuitive and convenient. Moreover, Dantzig's Simplex algorithm is still the most prevalent LP approach, while the MRM is historical.

stock types. For each stock type $k \in [K]$, Q^k is its set of cutting patterns and u^k is the number of available units. The objective is to produce the maximum number of articles. The symbols used in that description were adapted to match those in Formulation (2), but apart from that we now present the model in its original phrasing:

"We have the following conditions for the determination of the unknowns $\lambda^k_{\boldsymbol{q}}$:

1) $\lambda^k_{\boldsymbol{q}} \geq 0$ and equal to whole numbers;

2) $\sum_{\boldsymbol{q} \in Q^k} \lambda^k_{\boldsymbol{q}} = u^k$;

3) $$\frac{\sum_{k \in [K]} \sum_{\boldsymbol{q} \in Q^k} \boldsymbol{q}_1 \lambda^k_{\boldsymbol{q}}}{d_1} = \frac{\sum_{k \in [K]} \sum_{\boldsymbol{q} \in Q^k} \boldsymbol{q}_2 \lambda^k_{\boldsymbol{q}}}{d_2} = \ldots = \frac{\sum_{k \in [K]} \sum_{\boldsymbol{q} \in Q^k} \boldsymbol{q}_J \lambda^k_{\boldsymbol{q}}}{d_J}$$

and that their common value be a maximum."

Translating that to modern notation, we obtain:

$$\max \ z \tag{4a}$$

$$\text{s.t.} \quad \sum_{k \in [K]} \sum_{\boldsymbol{q} \in Q^k} \boldsymbol{q}_j \lambda^k_{\boldsymbol{q}} = d_j z \qquad j \in [J] \tag{4b}$$

$$\sum_{\boldsymbol{q} \in Q^k} \lambda^k_{\boldsymbol{q}} = u^k \qquad k \in [K] \tag{4c}$$

$$\boldsymbol{\lambda} \geq \mathbf{0} \quad \text{and integer,} \tag{4d}$$

where variable z represents the "common value", which is nothing but the number of articles produced. Kantorovich did not restrict that formulation to 1D cutting. On the contrary, several of the mentioned cases of use (page 379) are 2D cutting (sheets of glass or iron, boards, etc). By curiosity, the model is classified as having a Problem C structure. It is also clear that Kantorovich is assuming that the CSP instances that will be handled are small enough to permit the enumeration of all relevant cutting patterns in advance, there is no suggestion of CG.

The chapter proceeds by presenting "a very simple problem" that corresponds to the standard 1D CSP with a single stock type: how to cut 100 copies of each of three items with lengths 2.9, 2.1, and 1.5 using the minimum number of stocks of length 7.4? In other words, the instance $J = 3$, $w = (2.9 \ 2.1 \ 1.5)$, $d = (100 \ 100 \ 100)$, and $W = 7.4$. Six cutting patterns are enumerated and presented in a table that is reproduced in Figure 1. The optimal CSP solution, which is said to have been obtained by the MRM, is then shown: 30 stocks cut with cutting pattern I, 10 with II, and 50 with IV. Finally, a problem corresponding to an instance of his more general CSP model is posed: if an article requires one copy of each of three items, with lengths 2.9, 2.1, and 1.5, and there are 100 stocks of length 7.4 and 50 stocks of length 6.4, what is the maximum number of articles that can be produced? The optimal solution (producing 161 articles) is given. Incidentally, that problem is one of the examples

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TABLE 7

I	II	III	IV	V	VI
2.9	2.9	2.1	2.9	1.5	2.9
1.5	2.9	2.1	2.1	1.5	2.1
1.5	1.5	1.5	2.1	1.5	1.5
1.5		1.5		2.1	
7.4	7.3	7.2	7.1	6.6	6.5

Fig. 1 Table presenting six cutting patterns in [26]

used to illustrate the MRM in Appendix 1. So, on pages 407–408 one can find the detailed resolution process.

The misrepresentation of the contributions in [20] to the CSP began already in the 1960s. The first page of [18] has the following footnote: "The referee has kindly brought to our attention the even earlier work of L.V. Kantorovich, 'Mathematical Methods of Organizing and Planning Production', reprinted in Management Sci. 6, 366-422 (1962) [sic]". No further comments are made. In the subsequent article [19] we only find: "Specific examples were given by Kantorovich in his very early discussion of the trim problem". They did not realize that Kantorovich had proposed a formulation based on cutting patterns very similar to theirs. Of course, that acknowledgment would not remove the merits of Gilmore and Gomory, who not only arrived at it independently but also proposed its solution using CG, which is essential for handling large instances.

3 Linear programming banned in the USSR

This section merely compiles known historical facts and is based on [28], [16], [33], [39], [3], [4], and [14]. After publishing [20], Kantorovich was enthusiastic about linear programming [3]. He had very ambitious goals and wanted to use his methods not only on numerous local-level industrial problems but also for the central planning of the whole Soviet economy! He started to write an advanced manuscript with those ideas. Then, in June 1941, Nazi Germany invaded USSR. In September, Leningrad was already besieged, a siege that would last for 900 days. In 1942 he was evacuated from Leningrad and completed his manuscript named *The Best Use of Economic Resources*. He sent it to Gosplan, the powerful central economic planning agency. After its strong condemnation in 1943, he was forced to keep the manuscript unpublished. There were practical objections to Kantorovich's proposal. For example, it was argued that solving those large LPs would require vast human computational resources. However, the main

[3] Kantorovich was so comfortable with the technique that he was able to reformulate the classic optimal transport problem, originally proposed by Gaspard Monge in the 18th century, into an LP (which could be infinite-dimensional!) and apply duality theory. This approach made the problem significantly more tractable. That work [24] had a major impact across various fields, including economics, physics, and partial differential equations.

objections were ideological and related to the Labor Theory of Value: *the value of a good is 100% determined by the amount of labor required to produce it*. That theory of value is central to Marxism, which also affirms that the dissociation between price and value is the prime mechanism used in capitalism to exploit the working class.

– It was observed that dual variables (the conspicuous optimal resolving multipliers) may have a natural interpretation as prices. The most zealous Marxists found that highly problematic since they viewed prices (at least those that do not match labor value) as a harmful capitalist artifact that had to be eventually eliminated.

It can be said that those Marxists in Gosplan were not crazy. The then competing Marginalist Theory of Value (which is now the most widely accepted theory of value) states: *the value of a good is given by how much gain one additional unit of it brings*. One can argue that this is exactly the meaning of dual variables, which, because of that, are sometimes called *shadow prices* or *marginal prices*. In fact, linear programming would later become a major influence on Western economics (see for example the classic book [11]). Nevertheless, it is appalling that such a practical mathematical tool could be rejected, precisely at a moment when the USSR was in dire need of increasing its production, because of ideological subtleties.

Although unofficial, the ban on linear programming was dead serious [4]. Since their ideas could easily clash with Marxist orthodoxy, economists were among the most persecuted groups of intellectuals during Stalin's rule. Even world-famous economists were not spared. For example, Nikolai Kondratiev (1892–1938), the creator of the theory of long economic cycles, was imprisoned in a Gulag and later executed. As a result, Kantorovich would only dared to teach general linear programming at the Leningrad University in 1956, after Stalin's death! A version of *The Best Use of Economic Resources* was finally published in 1959 (it was translated to English as [27]). Emboldened by the less repressive environment of the period, he even advocated that *prices should be actively used in a socialist economy* as a tool for obtaining a more efficient allocation of resources.

Isolated uses of linear programming on particular problems occurred in the 1940s. According to [16], this was already happening during World War II: "Most of the work that Kantorovich did for the Soviet military remains classified to this day. We do know that Kantorovich applied his technique [LP] to the problem of cutting metal for tanks and to the problem of laying minefields." A few short publications on optimal cutting appeared at that time: [23] and [25].

4 A CG algorithm in Kantorovich and Zalgaller (1951)

A well-documented use of LP occurred in 1948–1949 on the cutting of metal at the Leningrad Egorov railway car building plant. The work was carried out mainly by Viktor A. Zalgaller (1920–2020), under the supervision of Kantorovich. The experience

[4] During the 1943 Gosplan meeting, one speaker said: "An optimum has already been proposed by the fascist Pareto, a favorite of Mussolini" (page 263 of [3]), a very threatening remark in that political context. There was even a closed-door discussion on whether it was necessary to arrest Kantorovich (page 433 of [14]).

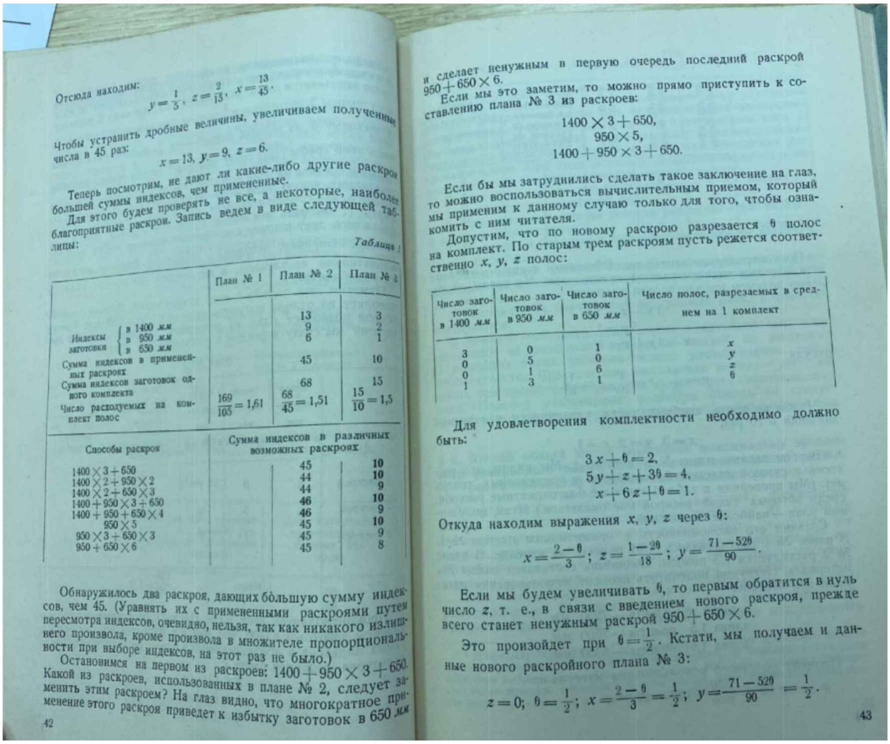
resulted in a 197-page book only on the CSP [21]. Appendix A presents a translation of its preface, in which Kantorovich himself describes the circumstances that led to the book and credits Zalgaller with several original ideas.

The main chapters of [21] are the following:

- Chapter 1: “General Methods for Solving the Cutting Problem”, pages 11–56. The chapter starts by motivating the importance of performing cutting in a rational way in order to reduce waste. Then, there is a discussion of “technological requisites”, which includes the issue of guillotine vs. non-guillotine 2D cutting. The chapter then presents the cutting pattern based LP models
  that will be used in the book and the overall solution method. The models are not viewed as IPs, so fractional use of a cutting pattern is acceptable. The assumption is that demands represent proportions.
  - In the provided example, an article requires 2 copies of item 1, 4 copies of item 2, and 1 copy of item 3, to be cut from a single stock type. The actual number of articles that will be manufactured is unknown, as the factory will be operated for an undetermined time.
  - So, the CSP is solved with demands $d = (\,2\;4\;1\,)$. Its fractional solution will determine the proportions in which each cutting pattern should be used. The optimal solution value is the average number of stocks used per article.

  The dual variables are called “indices”, possibly the most anodyne name that the authors could think of to avoid ideological controversies.
- Chapter 2: “Cutting Linear Materials to Length (rolled profiles, pipes, bars, strips)”, pages 57–108. It presents techniques for generating 1D cutting patterns. Even technical details on the Soviet machines of the time that could be used for performing the cuts are discussed.
- Chapter 3: “Cutting Sheet Material into Rectangular Items”, pages 109–170. The chapter presents techniques for generating 2D cutting patterns over rectangular stocks. It is mainly on cutting rectangular items, but it also considers circular items (pages 130–134) and even parallelogram items (pages 147–148). It includes big real examples of metal cutting from the Egorov plant, one with 63 rectangular items (pages 110-111) and another with 71 rectangular items (pages 152–155).

Unlike in [20], the cutting patterns are not assumed to be enumerated in advance. In fact, [21] proposes an iterative approach that can be regarded as a complete column generation algorithm. They propose finding improving patterns by what we now call reduced costs and state the optimality criterion. We reproduce here the steps for solving the 1D single stock type CSP instance having $J = 3$, $w = (\,1400\;950\;650\,)$, $d = (\,2\;4\;1\,)$, and $W = 5000$. We kept the notation very close to the original, except that here the dual variables (the “indices”) are denoted π_1, π_2, and π_3. The starting solution (page 40) only uses single-item patterns: $(\,3\;0\;0\,)$ with value 2/3, $(\,0\;5\;0\,)$ with value 4/5, $(\,0\;0\;7\,)$ with 1/7; the solution cost is ≈ 1.61. After two patterns are generated, the solution (page 41) is $(\,3\;0\;1\,)$ with value 2/3, $(\,0\;5\;0\,)$ with value 71/91, $(\,0\;1\;6\,)$ with value 1/18; the solution cost is ≈ 1.51. At that point, the indices are calculated



Отсюда находим:

$$y=\frac{1}{5},\ z=\frac{2}{15},\ x=\frac{13}{45}.$$

Чтобы устранить дробные величины, увеличиваем полученные числа в 45 раз:

$$x=13,\ y=9,\ z=6.$$

Теперь посмотрим, не дают ли какие-либо другие раскрои большей суммы индексов, чем примененные.

Для этого будем проверять не все, а некоторые, наиболее благоприятные раскрои. Запись ведем в виде следующей таблицы:

Таблица

	План № 1	План № 2	План № 3
Индексы заготовки в 1400 *мм*		13	3
Индексы заготовки в 950 *мм*		9	2
Индексы заготовки в 650 *мм*		6	1
Сумма индексов в примененных раскроях		45	10
Сумма индексов заготовок одного комплекта		68	15
Число расходуемых на комплект полос	$\frac{169}{105}=1,61$	$\frac{68}{45}=1,51$	$\frac{15}{10}=1,5$

Способы раскроя	Сумма индексов в различных возможных раскроях		
1400 × 3 + 650		45	10
1400 × 2 + 950 × 2		44	10
1400 × 2 + 650 × 3		44	9
1400 + 950 × 3 + 650		46	10
1400 + 950 + 650 × 4		46	9
950 × 5		45	10
950 × 3 + 650 × 3		45	9
950 + 650 × 6		45	8

Обнаружилось два раскроя, дающих бо́льшую сумму индексов, чем 45. (Уравнять их с примененными раскроями путем пересмотра индексов, очевидно, нельзя, так как никакого излишнего произвола, кроме произвола в множителе пропорциональности при выборе индексов, на этот раз не было.)

Остановимся на первом из раскроев: 1400 + 950 × 3 + 650. Какой из раскроев, использованных в плане № 2, следует заменить этим раскроем? На глаз видно, что многократное применение этого раскроя приведет к избытку заготовок в 650 *мм*

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и сделает ненужным в первую очередь последний раскрой 950 + 650 × 6.

Если мы это заметим, то можно прямо приступить к составлению плана № 3 из раскроев:

1400 × 3 + 650,
950 × 5,
1400 + 950 × 3 + 650.

Если бы мы затруднились сделать такое заключение на глаз, то можно воспользоваться вычислительным приемом, который мы применим к данному случаю только для того, чтобы ознакомить с ним читателя.

Допустим, что по новому раскрою разрезается θ полос на комплект. По старым трем раскроям пусть режется соответственно *x*, *y*, *z* полос:

Число заготовок в 1400 *мм*	Число заготовок в 950 *мм*	Число заготовок в 650 *мм*	Число полос, разрезаемых в среднем на 1 комплект
3	0	1	*x*
0	5	0	*y*
0	1	6	*z*
1	3	1	θ

Для удовлетворения комплектности необходимо должно быть:

$$3x+\theta=2,$$
$$5y+z+3\theta=4,$$
$$x+6z+\theta=1.$$

Откуда находим выражения *x*, *y*, *z* через θ:

$$x=\frac{2-\theta}{3};\ z=\frac{1-2\theta}{18};\ y=\frac{71-52\theta}{90}.$$

Если мы будем увеличивать θ, то первым обратится в нуль число *z*, т. е., в связи с введением нового раскроя, прежде всего станет ненужным раскрой 950 + 650 × 6.

Это произойдет при $\theta=\frac{1}{2}$. Кстати, мы получаем и данные нового раскройного плана № 3:

$$z=0;\ \theta=\frac{1}{2};\ x=\frac{2-\theta}{3}=\frac{1}{2};\ y=\frac{71-52\theta}{90}=\frac{1}{2}.$$

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Fig. 2 Photography of pages 42–43 of [21]

as the solution of the following 3 × 3 linear system (pages 41–42):

$$\begin{cases} 3\pi_1 \quad\; + \;\; \pi_3 = 1 \\ \quad 5\pi_2 \qquad\;\;\; = 1 \\ \quad\;\; \pi_2 + 6\pi_3 = 1 \end{cases} \Rightarrow \begin{aligned} \pi_1 &= 13/45 \\ \pi_2 &= 1/5 \\ \pi_3 &= 2/15. \end{aligned}$$

By solving an integer knapsack problem, the improving pattern (1, 3, 1) is found ($13/45 + 3/5 + 2/15 = 46/45 > 1$). Variables x, y, z are associated with the current patterns and θ to the new one, yielding (page 43, see Figure 2):

$$\begin{cases} 3x \qquad\quad + \;\; \theta = 2 \\ \quad 5y + \;\; z + 3\theta = 4 \\ x \qquad + 6z + \;\; \theta = 1 \end{cases} \Leftrightarrow \begin{cases} 3x \qquad\quad\;\; = 2 - \theta \\ \quad 5y + \;\; z = 4 - 3\theta \\ x \qquad + 6z = 1 - \theta \end{cases}$$

Solving the 3 × 3 linear system (considering θ as constants in the RHS), the following expressions are obtained:

$$x = \frac{2-\theta}{3}, \quad z = \frac{1-2\theta}{18}, \quad y = \frac{71-52\theta}{90}.$$

Therefore, when θ increases, the first value which nullifies is z (when $\theta = \frac{1}{2}$). Thus $(\,0\ 1\ 6\,)$ is replaced with $(\,1\ 3\ 1\,)$. It can be deduced that $x = \frac{1}{2}$ and $y = \frac{1}{2}$. The cost of the new solution is thus 1.5. Then, indices are recalculated by solving (page 44):

$$\begin{cases} 3\pi_1 \qquad\quad + \pi_3 = 1 \\ \qquad 5\pi_2 \qquad\ = 1 \\ \pi_1 + 3\pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \begin{aligned} \pi_1 &= 3/10 \\ \pi_2 &= 2/10 \\ \pi_3 &= 1/10. \end{aligned}$$

By solving another integer knapsack problem, it is shown that no improving pattern exists and therefore the current CSP solution is optimal. This means that patterns $(\,3\ 0\ 1\,)$, $(\,0\ 5\ 0\,)$, and $(\,1\ 3\ 1\,)$ should be used in equal proportions and each produced article will require on average 1.5 stocks.

Note that the proposed CG algorithm does not employ the Method of Resolving Multipliers. Instead, it has a logic that closely resembles the Revised Simplex Method [6]. A square "basis matrix" B, formed by the columns corresponding to J cutting patterns, is maintained throughout the iterations. The current basic primal solution x is defined by the system $Bx = d$. The associated dual solution is obtained by solving the linear system $\pi B = \mathbb{1}$ (the RHS vector corresponds to the unitary stock costs). A knapsack pricing problem may find a new variable with negative reduced cost. The basic solution is then updated by performing a "pivot": the new entering variable is increased, while the remaining basic variables are adjusted so as to preserve primal feasibility, until some basic variable reaches zero. A variable that first becomes zero leaves the basis. It should be emphasized that this CG algorithm was tailored to the specific structure of the CSP, whereas Dantzig's Revised Simplex Method was proposed for general LPs.

But how were the integer knapsack problems solved? Chapter 2 proposes the so-called *Scale of Indices Method*, which can be viewed as a graphical version of a Dynamic Programming algorithm. The optimal scale of indices corresponding to the last knapsack problem in the above example is shown in Figure 3 (a reproduction of Figure 8 on page 67 of [21]). The values of indices are multiplied by 10 to make them integers. The scale of indices indicates the best solution value for each knapsack capacity up to W, the solutions themselves are also indicated. As the value for $W = 5000$ is 10 (1.0 after dividing it by 10), it is shown that there is no improving pattern.

The optimal scale of indices is iteratively constructed using two sheets of paper, one of them being semi-transparent (the other may be regular paper), as illustrated in Figure 4. Two copies of the starting scale of indices (shown in (a)) should be plotted, one on each sheet. The starting scale should have the values corresponding to single-item solutions (1 for capacity 650, 2 for capacity 950, and 3 for capacity 1400), plus some possibly heuristic non-optimal values for larger values of capacity. Then the copy at the bottom (the one on regular paper) is shifted forwards and upwards (or equivalently, the copy at the top, the one on semi-transparent paper, is shifted backward and downwards), as illustrated in (b), a reproduction of Figure 9 in page 68 of [21]. The dashed regions indicate better knapsack solutions. Those improvements are marked in the transparent paper at the top, leading to the improved scale shown in (c). The procedure is repeated (the starting scale on regular paper can always be kept at the bottom) until no improvement is possible.

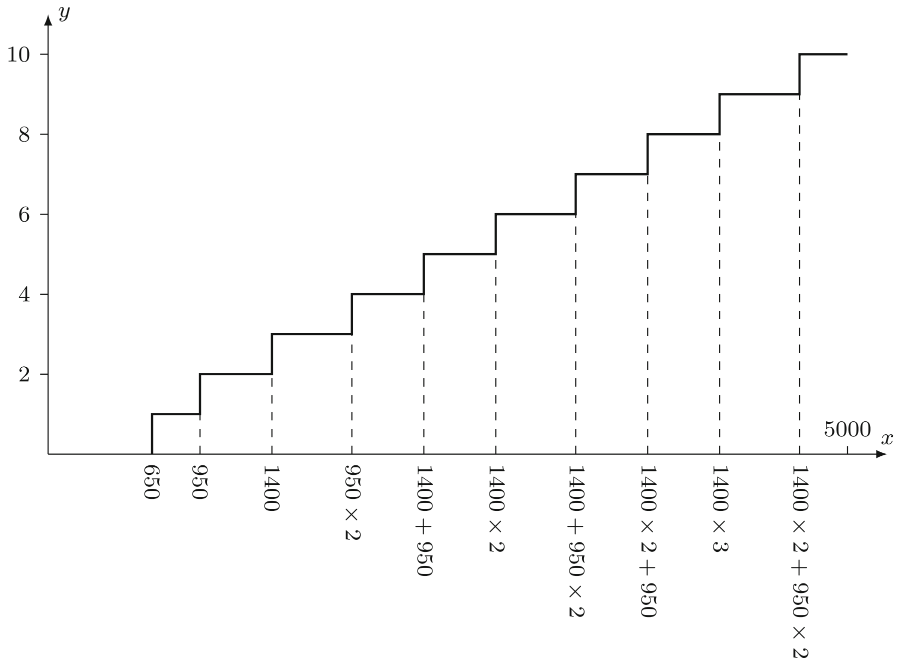

Fig. 3 An optimal "Scale of Indices"

In today's context, the approach seems odd. However, in the era before computers, it was a widespread practice for engineers to utilize mechanical analog tools, such as slide rules, to speed up computations. Due to its parallel structure, the Scale of Indices method is capable of evaluating several potential improvements simultaneously, convergence is usually fast. Nonetheless, similar to most mechanical analog techniques, the method suffers from low numerical precision. In contrast, the Dynamic Programming knapsack algorithm with explicit stage-by-stage numerical calculations proposed by Richard Bellman [1] can have arbitrary precision.

The successful implementation of the cutting optimization methods at the Egorov plant had unexpected consequences. According to [28], it greatly reduced the amount of metal scrap produced. Since some nearby steel plants relied on this scrap as a primary input, Kantorovich was reportedly ordered to appear at Leningrad party headquarters, accused of sabotaging the local economy. He was allegedly rescued by the intervention of the military, which required his expertise for the ongoing atomic program. This anecdote may be somewhat exaggerated, perhaps serving as a cautionary tale about the dangers of innovating in Soviet Union. Indeed, [34] offers a different account, only stating that the plant's manager received a reprimand for "failing to meet the scrap quota".

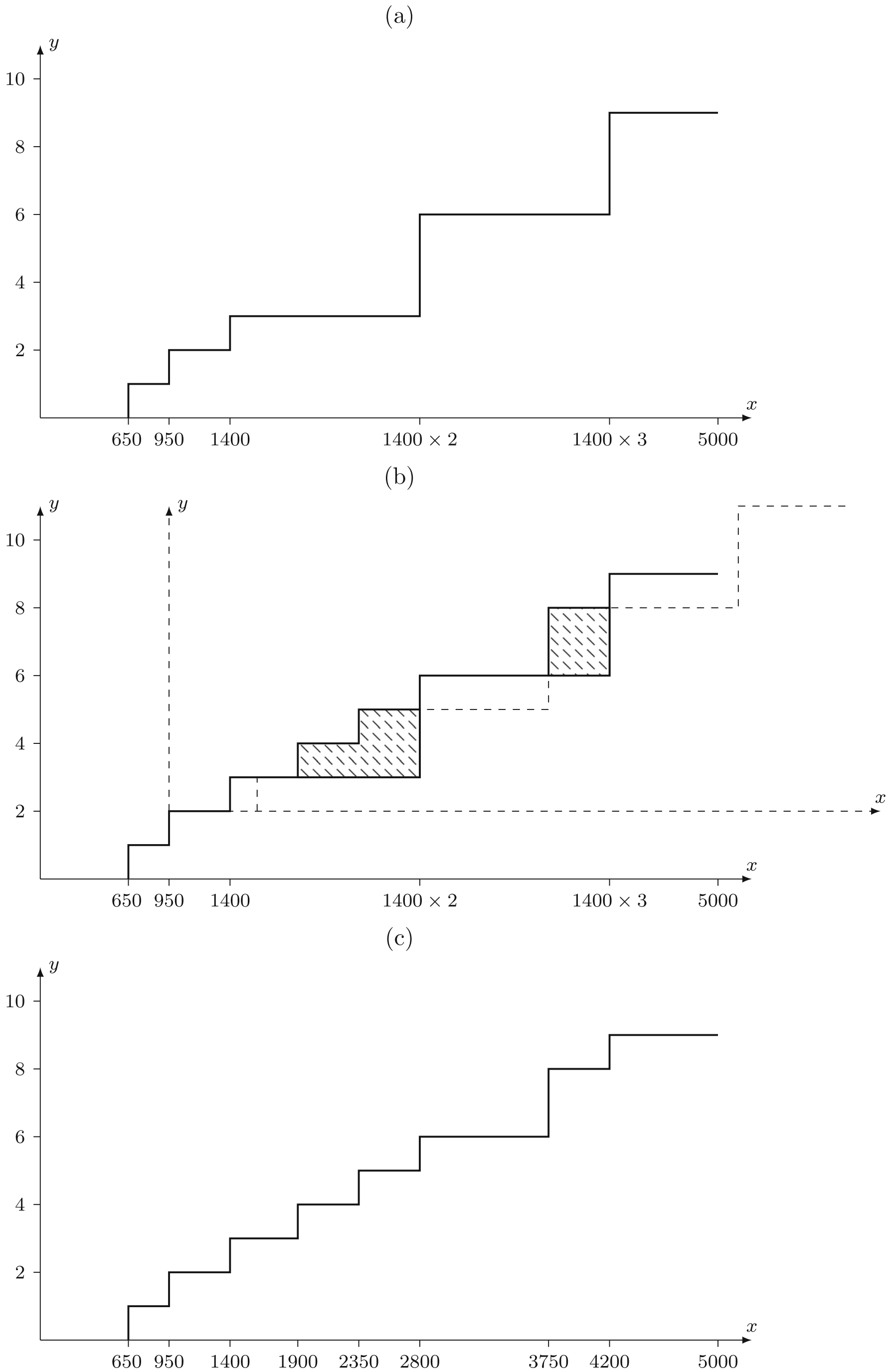

Fig. 4 The initial scale of indices is shown on (a). In (b), a copy of that scale shifted by (950, 2) obtains the improvements depicted as dashed regions. The resulting improved scale is shown in (c). The procedure should be repeated until no improvement is possible

5 Conclusions

The material of this article was presented as a talk in the Column Generation Workshop, an event that happened in May 2023 in Montreal, and gathered most of the experts in the field. No one in that audience was aware of the true contents of [20] on the CSP or that [21] contains a CG algorithm (actually, most of them never even heard about that book). Therefore, we believe that we are bringing a valuable contribution to the history of our field.

- Based on the findings in Section 2, we propose that the cutting pattern based Formulation (2) of the CSP, often referred to as the Gilmore-Gomory formulation, should be renamed as the Kantorovich-Gilmore-Gomory formulation.
- Based on the findings in Section 4, we may say that in the late 1940s Soviet scientists (despite the mind-boggling ideological restrictions mentioned in Section 3) were far ahead of their Western counterparts on handling CSPs [5], as much in theory as in practice, even anticipating the CG technique.

The Western ignorance of [21] is hardly surprising in the context of a Cold War that limited the exchanges with the Communist bloc countries. Language barriers were then much higher, too. There are other cases of major discoveries made in one bloc that remained unknown in the other bloc for many years. For example, the Affine Scaling interior-point LP algorithm was discovered by Soviet mathematician I.I. Dikin in 1967 and reinvented in the US (by three independent groups) in the mid-1980s. Note that [10] was not an obscure article. Much to the contrary, it was presented by Kantorovich himself (it was not rare for Soviet articles to be officially presented by a senior colleague, as a kind of endorsement of its contents) and published in the USSR equivalent of the US Proceedings of the National Academy of Sciences.

Similarly, [21] (5,000 printed copies) was an influential book within the Soviet Union. His first author was a celebrity, having received the 1949 Stalin Prize [6], the highest Soviet scientific honor. The book was followed by several other Soviet works on the topic (see [34]), including a monograph by Zalgaller on log cutting [42]. A computer implementation of its CG algorithm for 1D cutting appears in [2]. The book was popular enough to deserve a second edition [22] (4,600 copies) that includes [42] as an additional chapter. *Yet, it had a minimum impact outside the Soviet bloc, being almost unknown in the Western world until today*. In particular, the proposed CG algorithm does not seem to have influenced the mainstream development of the field. This is why we call it *The 0-th Column Generation Algorithm*.

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[5] The first Western article presenting a mathematical treatment of the CSP seems to be [13].

[6] The prize was also a reward for his (secret of the time) role as the chief mathematical calculator in the Soviet atom bomb program (page 429 of [14]).

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A Preface of Kantorovich and Zalgaller (1951)

We provide an unabridged translation of the preface of [21]:

"*The great practical importance of the issue of rational cutting of industrial materials as an important source of cost savings in production has been repeatedly noticed in technical literature and journal articles.*

However, on the scientific (theoretical) side, this issue has been developed very little. Here we can name the well-known problem of the tightest arrangement of circles on a plane, equivalent to the question of cutting a large sheet into circles, as well as some other problems of a similar nature from the field of discrete geometry, which has limited practical significance. A peculiar and subtle investigation devoted to the cutting of materials belongs to the great Russian mathematician P. L. Chebyshev (, see reference below), but it does not deal with the question of the most economical cutting, which is the subject of this work, but with the problem of the most accurate covering of a curved surface by flat fabric cuttings.*

Finally, we could mention some works related to maximizing yield in cutting specifically in the field of sawmilling.

The definition and some general approach to the analysis of the question of rational cutting were given in my work of 1939, which deals with production questions of various natures, in which it is required to choose the best solution among many possible ones. The use of the general method of resolving multipliers, developed in this work, in application to the question of cutting gives a characterization of the optimal cutting plan and proves that it is possible to find it. The question of cutting has been further developed in several other works of mine.

In 1948-1949 in the Leningrad branch of the Mathematical Institute of the USSR Academy of Sciences, we set the task of more detailed development of these methods and their practical testing at Leningrad enterprises. This work was carried out, under my general supervision, by V. A. Zalgaller, a researcher at the institute.

The main site for the realization of this work was chosen as the Leningrad Egorov railway car building plant, where metal is consumed in large quantities in the production of all-metal railway cars.

A number of employees of this plant, in particular employees of the department of the chief technologist (head of the department G. A. Treubov), as well as foremen and workers, took an active part in the implementation of these methods in a production *environment. Thus, this book is an original result of the creative collaboration of mathematical scientists and industrial workers.*

It should be said that although in the process of this work it was found out that the method of resolving multipliers (indices) was very useful in solving factory problems, it had to be developed and adapted to industrial problems and supplemented with essentially new solutions and technical methods. Among them, we should mention new solution methods developed by V. A. Zalgaller: selection of integer indices, analysis of Problem 2 (Chapter 1, Section 2), solution of a planar problem by means of an auxiliary linear problem, substantially developed by him methods of cutting materials of mixed lengths, in particular, the theory of construction of a measuring ruler (see Appendix 2), and technical adaptations proposed by him: use of a sorting rack, adaptation of the ruler to the machine. Finally, he has developed a practical methodology for the use of the whole set of working techniques (sequence of calculation, selection of the appropriate method, consideration of technological requirements, necessary organizational measures, documentation, etc.).

In addition to the techniques developed recently in solving practical problems for the Egorov plant and some other enterprises, the book utilizes the previously mentioned materials; finally, some issues were developed by the authors in the very process of writing the book.

The text of the book was written mainly by V.A. Zalgaller according to the plan drawn up by both authors. I have done mainly editorial work on it.

This book, which combines all the accumulated material and experience, is intended to familiarize engineers and technical workers of enterprises with the proposed methods of computing the most rational cutting plans to ensure the possibility of widespread dissemination of these methods at enterprises.

The book is intended primarily for technologists of groups of material standards and procurement shops of machine-building enterprises.

Prof. L. V. Kantorovich

(*) P. L. Chebyshev, About dress cutting, journal "Advances in Mathematics", 1(9):38-42 (1946), in Russian. (The manuscript is dated 1878.)"

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